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# Triple cohomology of Lie–Rinehart algebras and the canonical class of associative algebras

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**Abstract**

We introduce a bicomplex which computes the triple cohomology of Lie–Rinehart algebras. We prove that the triple cohomology is isomorphic to the Rinehart cohomology provided the Lie–Rinehart algebra is projective over the corresponding commutative algebra. As an application we construct a canonical class in the third dimensional cohomology corresponding to an associative algebra and extend Sridharan's result on almost commutative algebras.


*Keywords:* Lie–Rinehart algebra; Hochschild cohomology; Cotriple

## 1. Introduction

Let A be a commutative algebra over a field $K$. A Lie–Rinehart algebra is a Lie $K$-algebra, which is also an A-module and these two structures are related in an appropriate way [7]. The leading example of Lie–Rinehart algebras is the set Der(A) of all $K$-derivations of A. Lie–Rinehart algebras are algebraic counterpart of Lie algebroids [11].

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The cohomology $H^*_{\mathrm{Rin}}(\mathcal{L}, M)$ of a Lie–Rinehart algebra $\mathcal{L}$ with coefficients in a Lie–Rinehart module $M$ was first defined by Rinehart [14] and further developed by Huebschmann [7]. However these groups have good properties only in the case when $\mathcal{L}$ is projective over A. In this paper following to [13] we introduce a bicomplex $C^{**}(\mathrm{A}, \mathcal{L}, M)$, whose cohomology $H^*(\mathrm{A}, \mathcal{L}, M)$ is isomorphic to $H^*_{\mathrm{Rin}}(\mathcal{L}, M)$ provided $\mathcal{L}$ is projective as an A-module. It turns out, that for general $\mathcal{L}$ the group $H^*(\mathrm{A}, \mathcal{L}, M)$ is isomorphic to a triple cohomology of Barr–Beck [1] applied to Lie–Rinehart algebras. We also prove that for general $\mathcal{L}$, unlike the Rinehart cohomology $H^*_{\mathrm{Rin}}(\mathcal{L}, M)$, the groups $H^*(\mathrm{A}, \mathcal{L}, M)$ in dimensions two and three classify all abelian and crossed extensions of $\mathcal{L}$ by $M$.

It should be mentioned that the cohomology groups $H^*(\mathrm{A}, \mathcal{L}, M)$ are new even for Lie algebras. The classical theory of Chevalley–Eilenberg works well only in the case when a Lie algebra $\mathcal{L}$ is projective as a module over the ground algebra A. The recent work of Barr [2] shows that in this case the classical theory defined via Chevalley–Eilenberg complex is isomorphic to a cotriple cohomology of Barr and Beck. Therefore, our result extends Barr's not only to all Lie algebras, but also to all Lie–Rinehart algebras as well.

The fact that $H^2(\mathrm{A}, \mathcal{L}, M)$ classifies all abelian extensions of Lie–Rinehart algebras is used to classify almost commutative algebras, such that the associated graded algebra is isomorphic to a symmetric algebra over A on a free A-module. These results extend the result of Sridharan, who considered the case $\mathrm{A} = K$.

The fact that $H^3(\mathrm{A}, \mathcal{L}, M)$ classifies all crossed extensions of Lie–Rinehart algebras is used to construct a canonical class corresponding to an associative algebra $S$. This construction uses the Hochschild cohomology of $S$ with coefficients in $S$, which is denoted by $H^*(S, S)$. It is well known that $H^1(S, S)$ is a Lie $K$-algebra. It turns out that $H^1(S, S)$ is in fact a Lie–Rinehart algebra over A, where $\mathrm{A} = H^0(S, S)$ is the center of $S$. Thus we can consider the cohomology $H^*(\mathrm{A}, H^1(S, S), \mathrm{A})$. We construct an element

$$o(S) \in H^3\big(\mathrm{A}, H^1(S, S), \mathrm{A}\big)$$

which we call the *canonical class of* $S$. $o(S)$ measures the noncommutativity of $S$ and we prove that $o(S)$ is a Morita invariant. The construction of $o(S)$ uses crossed modules of Lie–Rinehart algebras introduced in [4].

## 2. Preliminaries on Lie–Rinehart algebras

The material of this section is well known. We included it in order to fix terminology, notations and main examples. In what follows we fix a field $K$. All vector spaces are considered over $K$. We write $\otimes$ and $\mathsf{Hom}$ instead of $\otimes_K$ and $\mathsf{Hom}_K$.

### 2.1. Definitions, examples

Let A be a commutative algebra over a field $K$. Then the set Der(A) of all $K$-derivations of A is a Lie $K$-algebra and an A-module simultaneously. These two structures are related by the following identity

$$\big[D, aD'\big] = a\big[D, D'\big] + D(a)D', \quad D, D' \in \mathrm{Der(A)}.$$

This leads to the notion below, which goes back to Herz under the name “pseudo-algèbre de Lie” (see [6]) and which is algebraic counterpart of the Lie algebroid [11].

**Definition 2.1.** A *Lie–Rinehart algebra* over A consists of a Lie $K$-algebra $\mathcal{L}$ together with an A-module structure on $\mathcal{L}$ and a map

$$\alpha:\mathcal{L}\to \mathrm{Der}(\mathrm{A})$$

which is simultaneously a Lie algebra and A-module homomorphism such that

$$[X,aY]=a[X,Y]+X(a)Y.$$

Here $X,Y\in\mathcal{L}$, $a\in \mathrm{A}$ and we write $X(a)$ for $\alpha(X)(a)$ [7]. These objects are also known as $(K,\mathrm{A})$-Lie algebras [14] and $d$-Lie rings [12].

Thus Der(A) with $\alpha=\mathsf{Id}_{\mathrm{Der}(\mathrm{A})}$ is a Lie–Rinehart A-algebra. Let us observe that Lie–Rinehart A-algebras with trivial homomorphism $\alpha:\mathcal{L}\to \mathrm{Der}(\mathrm{A})$ are exactly Lie A-algebras. Therefore the concept of Lie–Rinehart algebras generalizes the concept of Lie A-algebras. If $\mathrm{A}=K$, then $\mathrm{Der}(\mathrm{A})=0$ and there is no difference between Lie and Lie–Rinehart algebras. We denote by $\mathcal{LR}(\mathrm{A})$ the category of Lie–Rinehart algebras. We have the full inclusion

$$\mathcal{L}(\mathrm{A})\subset\mathcal{LR}(\mathrm{A}),$$

where $\mathcal{L}(\mathrm{A})$ denotes the category of Lie A-algebras. Let us observe that the kernel of any Lie–Rinehart algebra homomorphism is a Lie A-algebra.

**Example 2.2.** If $\mathfrak{g}$ is a $K$-Lie algebra acting on a commutative $K$-algebra A by derivations (that is, a homomorphism of Lie $K$-algebras $\gamma:\mathfrak{g}\to \mathrm{Der}(\mathrm{A})$ is given), then *the transformation* Lie–Rinehart algebra of $(\mathfrak{g},\mathrm{A})$ is $\mathcal{L}=A\otimes\mathfrak{g}$ with the Lie bracket

$$\left[a\otimes g,a'\otimes g'\right]:=aa'\otimes\left[g,g'\right]+a\gamma(g)\left(a'\right)\otimes g'-a'\gamma\left(g'\right)(a)\otimes g$$

and with the action $\alpha:\mathcal{L}\to \mathrm{Der}(\mathrm{A})$ given by $\alpha(a\otimes g)(a')=a\gamma(g)(a')$.

**Example 2.3.** Let us recall that a *Poisson algebra* is a commutative $K$-algebra $P$ equipped with a Lie $K$-algebra structure such that the following identity holds

$$[a,bc]=b[a,c]+[a,b]c.$$

There are (at least) three Lie–Rinehart algebra related to $P$. The first one is $P$ itself considered as a $P$-module in an obvious way, where the action of $P$ (as a Lie algebra) on $P$ (as a commutative algebra) is given by the homomorphism $\mathsf{ad}:P\to \mathrm{Der}(P)$ given by

$$\mathsf{ad}(a)=[a,-]\in \mathrm{Der}(P).$$

The second Lie–Rinehart algebra is the Kähler differentials $\Omega^1_P$. It is easily shown (see [7]) that there is a unique Lie–Rinehart algebra structure on $\Omega^1_P$ such that $[da, db] = d[a, b]$ and such that the Lie algebra homomorphism $\Omega^1_P \to \mathrm{Der}(P)$ is given by $adb \mapsto a[b, -]$. To describe the third one, we need some preparations. We put

$$H^0_{\mathrm{Poiss}}(P, P) := \bigl\{a \in P \mid [a, -] = 0\bigr\}.$$

Then $H^0_{\mathrm{Poiss}}(P, P)$ contains the unit of $P$ and is closed with respect to products, thus it is a subalgebra of $P$. A *Poisson derivation* of $P$ is a linear map $D : P \to P$ which is a simultaneous derivation with respect to commutative and Lie algebra structures. We let $\mathsf{Der}_{\mathrm{Poiss}}(P)$ be the collection of all Poisson derivations of $P$. It is closed with respect to Lie bracket. Moreover if $a \in H^0_{\mathrm{Poiss}}(P, P)$ and $D \in \mathsf{Der}_{\mathrm{Poiss}}(P)$ then $aD \in \mathsf{Der}_{\mathrm{Poiss}}(P)$. It follows that $\mathsf{Der}_{\mathrm{Poiss}}(P)$ is a Lie–Rinehart algebra over $H^0_{\mathrm{Poiss}}(P, P)$. There is the following variant of the first construction in the graded case. Let $P_* = \bigoplus_{n\geqslant 0} P_n$ be a commutative graded $K$-algebra in the sense of commutative algebra (i.e., no signs are involved) and assume $P_*$ is equipped with a Poisson algebra structure such that the bracket has degree $(-1)$. Thus $[-, -] : P_n \otimes P_m \to P_{n+m-1}$. Then $P_1$ is a Lie–Rinehart $P_0$-algebra, where the Lie algebra homomorphism $P_1 \to \mathrm{Der}(P_0)$ is given by $a_1 \mapsto [a_1, -]$, $[a_1, -](a_0) = [a_1, a_0]$, where $a_i \in P_i$, $i = 0, 1$.

**Definition 2.4.** A *Lie–Rinehart module* over a Lie–Rinehart A-algebra $\mathcal{L}$ is a vector space $M$ together with two operations

$$\mathcal{L} \otimes M \to M, \quad (X, m) \mapsto X(m),$$

and

$$\mathrm{A} \otimes M \to M, \quad (a, m) \mapsto am,$$

such that the first one makes $M$ into a module over the Lie $K$-algebra $\mathcal{L}$ in the sense of the Lie algebra theory, while the second map makes $M$ into an A-module and additionally the following compatibility conditions hold

$$(aX)(m) = a\bigl(X(m)\bigr),$$
$$X(am) = aX(m) + X(a)m.$$

Here $a \in \mathrm{A}$, $m \in M$ and $X \in \mathcal{L}$.

It follows that A is a Lie–Rinehart module over $\mathcal{L}$ for any Lie–Rinehart algebra $\mathcal{L}$. We let $(\mathcal{L}, \mathrm{A})$-**mod** be the category of Lie–Rinehart modules over $\mathcal{L}$.

### 2.2. Rinehart cohomology of Lie–Rinehart algebras

Let $M$ be a Lie–Rinehart module over $\mathcal{L}$. Let us recall the definition of the Rinehart cohomology $H^*_{\mathrm{Rin}}(\mathcal{L}, M)$ of a Lie–Rinehart algebra $\mathcal{L}$ with coefficients in a Lie–Rinehart module $M$ (see [7,14]). We write

$$C^n_{\mathrm{A}}(\mathcal{L}, M) := \mathrm{Hom}_{\mathrm{A}}\left(\Lambda^n_{\mathrm{A}}\mathcal{L}, M\right),$$

where $\Lambda^*_{\mathrm{A}}(V)$ denotes the exterior algebra over A generated by an A-module $V$. The coboundary map

$$\delta : C^{n-1}_{\mathrm{A}}(\mathcal{L}, M) \to C^n_{\mathrm{A}}(\mathcal{L}, M)$$

is given by

$$\begin{aligned}(\delta f)(X_1, \ldots, X_n) = (-1)^n \sum_{i=1}^{n} (-1)^{(i-1)} X_i\big(f\big(X_1, \ldots, \hat{X}_i, \ldots, X_n\big)\big) \\ + (-1)^n \sum_{i<j} (-1)^{i+j} f\big([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_n\big).\end{aligned}$$

Here $X_1, \ldots, X_n \in \mathcal{L}$, $f \in C^{n-1}_{\mathrm{A}}(\mathcal{L}, M)$. By the definition $H^*_{\mathrm{Rin}}(\mathcal{L}, M)$ is the cohomology of the cochain complex $C^*_{\mathrm{A}}(\mathcal{L}, M)$. We observe that if $\mathrm{A} = K$, then this definition generalizes the classical definition of Lie algebra cohomology. For a general A by forgetting the A-module structure we obtain the canonical homomorphism

$$H^*_{\mathrm{Rin}}(\mathcal{L}, M) \to H^*_{\mathrm{Lie}}(\mathcal{L}, M),$$

where $H^*_{\mathrm{Lie}}(\mathcal{L}, M)$ denotes the cohomology of $\mathcal{L}$ considered as a Lie $K$-algebra. On the other hand if A is a smooth commutative algebra, then $H^*_{\mathrm{Rin}}(\mathrm{Der}(\mathrm{A}), \mathrm{A})$ is isomorphic to the de Rham cohomology of A (see [7,14]).

It follows from the definition that we have the following exact sequence

$$0 \to H^0_{\mathrm{Rin}}(\mathcal{L}, M) \to M \to \mathrm{Der}_{\mathrm{A}}(\mathcal{L}, M) \to H^1_{\mathrm{Rin}}(\mathcal{L}, M) \to 0, \tag{1}$$

where $\mathrm{Der}_{\mathrm{A}}(\mathcal{L}, M)$ consists of A-linear maps $d : \mathcal{L} \to M$ which are derivations from the Lie $K$-algebra $\mathcal{L}$ to $M$. In other words $d$ must satisfy the following conditions:

$$\begin{aligned}d(aX) &= ad(X), \quad a \in \mathrm{A},\ X \in \mathcal{L},\\ d\big([X, Y]\big) &= X\big(d(Y)\big) - Y\big(d(X)\big).\end{aligned}$$

For a Lie–Rinehart module $M$ over a Lie–Rinehart algebra $\mathcal{L}$ we can define the *semi-direct product* $\mathcal{L} \rtimes M$ to be $\mathcal{L} \oplus M$ as an A-module with the bracket $[(X, m), (Y, n)] = ([X, Y], X(n) - Y(m))$.

**Lemma 2.5.** *Let $\mathcal{L}$ be a Lie–Rinehart algebra over a commutative algebra* A *and let* $M \in (\mathcal{L}, \mathrm{A})$-**mod**. *Then there is a* 1–1 *correspondence between the elements of* $\mathrm{Der}_{\mathrm{A}}(\mathcal{L}, M)$ *and the sections* (*in the category* $\mathcal{LR}(\mathrm{A})$) *of the projection* $p : \mathcal{L} \rtimes M \to \mathcal{L}$.

**Proof.** Any section $\xi : \mathcal{L} \to \mathcal{L} \rtimes M$ of $p$ has the form $\xi(x) = (x, f(x))$ and it is easily shown that $\xi$ is a morphism in $\mathcal{LR}(\mathrm{A})$ iff $f \in \mathrm{Der}_{\mathrm{A}}(\mathcal{L}, M)$. □

*2.3. Abelian and crossed extensions of Lie–Rinehart algebras*

**Definition 2.6.** Let $\mathcal{L}$ be a Lie–Rinehart algebra over a commutative algebra A and let $M \in (\mathcal{L}, \mathrm{A})$-**mod**. An abelian extension of $\mathcal{L}$ by $M$ is an exact sequence

$$0 \to M \xrightarrow{i} \mathcal{L}' \xrightarrow{\partial} \mathcal{L} \to 0$$

where $\mathcal{L}'$ is a Lie–Rinehart algebra over A and $\partial$ is a Lie–Rinehart algebra homomorphism. Moreover $i$ is an A-linear map and the following identities hold:

$$\big[i(m), i(n)\big] = 0,$$
$$\big[i(m), X'\big] = \big(\partial\big(X'\big)\big)(m),$$

where $m, n \in M$ and $X' \in \mathcal{L}'$. An abelian extension is called A-split if $\partial$ has an A-linear section.

We also need the notion of crossed modules for Lie–Rinehart algebras introduced in [4]. The following definition is equivalent to the one given in [4].

**Definition 2.7.** A crossed module $\partial : \mathcal{R} \to \mathcal{L}$ of Lie–Rinehart algebras over A consists of a Lie–Rinehart algebra $\mathcal{L}$ and a Lie–Rinehart module $\mathcal{R}$ over $\mathcal{L}$ together with an A-linear homomorphism $\partial : \mathcal{R} \to \mathcal{L}$ such that for all $r, s \in \mathcal{R}$, $X \in \mathcal{L}$, $a \in \mathrm{A}$ the following identities hold:

(1) $\partial(X(r)) = [X, \partial(r)]$,
(2) $(\partial(r))(s) + (\partial(s))(r) = 0$,
(3) $\partial(r)(a) = 0$.

It follows from this definition that $\mathcal{R}$ is a Lie A-algebra under the bracket $[r, s] = (\partial(r))(s)$ and $\partial$ is a homomorphism of Lie $K$-algebras. Moreover $\mathsf{Im}(\partial)$ is simultaneously a Lie $K$-ideal of $\mathcal{L}$ and an A-submodule, therefore $\mathsf{Coker}(\partial)$ is a Lie–Rinehart algebra. Furthermore $\mathsf{Ker}(\partial)$ is an abelian A-ideal of $\mathcal{R}$ and the action of $\mathcal{L}$ on $R$ yields a Lie–Rinehart module structure of $\mathsf{Coker}(\partial)$ on $\mathsf{Ker}(\partial)$.

Let $\mathcal{P}$ be a Lie–Rinehart algebra and let $M$ be a Lie–Rinehart module over $\mathcal{P}$. We consider the category **Cross**$(\mathcal{P}, M)$, whose objects are the exact sequences

$$0 \to M \to \mathcal{R} \xrightarrow{\partial} \mathcal{L} \xrightarrow{\upsilon} \mathcal{P} \to 0$$

where $\partial : \mathcal{R} \to \mathcal{L}$ is a crossed module of Lie–Rinehart algebras over A and the canonical maps $\mathsf{Coker}(\partial) \to \mathcal{P}$ and $M \to \mathsf{Ker}(\partial)$ are isomorphisms of Lie–Rinehart algebras and modules respectively. The morphisms in the category $\mathbf{Cross}(\mathcal{P}, M)$ are commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & \mathcal{R} \arrow[r, "\partial"] \arrow[d, "\alpha"'] & \mathcal{L} \arrow[r] \arrow[d, "\beta"] & \mathcal{P} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & \mathcal{R}' \arrow[r, "\partial'"] & \mathcal{L}' \arrow[r] & \mathcal{P}' \arrow[r] & 0 \end{tikzcd}$$

where $\beta$ is a homomorphism of Lie–Rinehart algebras, $\alpha$ is a morphism of Lie A-algebras and for any $r \in \mathcal{R}$, $X \in \mathcal{L}$ we have

$$\alpha\big(X(r)\big) = \big(\beta(X)\big)\big(\alpha(r)\big).$$

Furthermore, we let $\mathbf{Cross}_{\text{A-spl}}(\mathcal{P}, M)$ be the subcategory of $\mathbf{Cross}(\mathcal{P}, M)$ whose objects and morphisms split in the category of A-modules, in other words, we require that the epimorphisms $\mathcal{L} \to \mathcal{P}$, $\mathcal{R} \to \mathsf{Im}(\partial)$, $\mathcal{L}' \to \mathcal{P}'$, $\mathcal{R}' \to \mathsf{Im}(\partial)'$, $\mathcal{L} \to \mathsf{Im}(\beta)$, $\mathcal{L}' \to \mathsf{Coker}(\beta)$, $\mathcal{R} \to \mathsf{Im}(\alpha)$, $\mathcal{R}' \to \mathsf{Coker}(\alpha)$ have A-linear sections.

### *2.4. Main properties of Rinehart cohomologies*

**Theorem 2.8.**

(i) *If $\mathcal{L}$ is projective as an* A*-module, then*

$$H^*_{\mathrm{Rin}}(\mathcal{L}, M) \cong \mathsf{Ext}^*_{(\mathcal{L},\mathrm{A})\text{-}\mathbf{mod}}(\mathrm{A}, M).$$

(ii) *If $0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence in the category* $(\mathcal{L}, \mathrm{A})$-**mod***, then we have a long exact sequence on cohomology*

$$\cdots \to H^n_{\mathrm{Rin}}(\mathcal{L}, M_1) \to H^n_{\mathrm{Rin}}(\mathcal{L}, M) \to H^n_{\mathrm{Rin}}(\mathcal{L}, M_2) \to \cdots$$

*provided $0 \to M_1 \to M \to M_2 \to 0$ splits in the category of* A*-modules or $\mathcal{L}$ is projective as an* A*-module.*

(iii) *The cohomology $H^2_{\mathrm{Rin}}(\mathcal{L}, M)$ classifies the abelian extensions*

$$0 \to M \to \mathcal{L}' \to \mathcal{L} \to 0$$

*of $\mathcal{L}$ by $M$ in the category of Lie–Rinehart algebras which split in the category of* A*-modules.*

(iv) *For any Lie–Rinehart algebra $\mathcal{P}$ which is projective as an* A*-module and any Lie–Rinehart module $M$ there exists a natural bijection between the classes of the connected components of the category $\mathbf{Cross}_{\text{A-spl}}(\mathcal{P}, M)$ and $H^3_{\mathrm{Rin}}(\mathcal{P}, M)$.*

**Proof.** For the isomorphism of the part (i) see [14, Section 4]. The part (ii) is trivial and for part (iii) see [7, Theorem 2.6]. Finally the part (iv), which is in the same spirit as the classical result for group and Lie algebra cohomology (see [8,9]), was proved in [4]. □

Let $\mathfrak{g}$ be a Lie algebra over $K$ and let $M$ be a $\mathfrak{g}$-module. Then we have the Chevalley–Eilenberg cochain complex $C^*_{\mathrm{Lie}}(\mathfrak{g}, M)$, which computes the Lie algebra cohomology (see [3]):

$$C^n_{\mathrm{Lie}}(\mathfrak{g}, M) = \mathsf{Hom}\big(\Lambda^n(\mathfrak{g}), M\big).$$

Here $\Lambda^*$ denotes the exterior algebra defined over $K$.

**Lemma 2.9.** *Let $\mathfrak{g}$ be a Lie $K$-algebra acting on a commutative algebra* A *by derivations and let $\mathcal{L}$ be the transformation Lie–Rinehart algebra of* $(\mathfrak{g}, \mathrm{A})$ (*see Example* 2.2). *Then for any Lie–Rinehart $\mathcal{L}$-module $M$ we have the canonical isomorphism of cochain complexes $C^*_{\mathrm{A}}(\mathcal{L}, M) \cong C^*_{\mathrm{Lie}}(\mathfrak{g}, M)$ and in particular the isomorphism*

$$H^*_{\mathrm{Rin}}(\mathcal{L}, M) \cong H^*_{\mathrm{Lie}}(\mathfrak{g}, M).$$

**Proof.** Since $\mathcal{L} = \mathrm{A} \otimes \mathfrak{g}$ we have $\mathrm{Hom}_{\mathrm{A}}(\Lambda^n_{\mathrm{A}}\mathcal{L}, M) \cong \mathsf{Hom}(\Lambda^n\mathfrak{g}, M)$ and lemma follows. □

## 3. The main construction

Thanks to Theorem 2.8 the cohomology theory $H^*_{\mathrm{Rin}}(\mathcal{L}, -)$ has good properties only if $\mathcal{L}$ is projective as an A-module. In this section we introduce the bicomplex $C^{**}(\mathrm{A}, \mathcal{L}, M)$, whose cohomology is a good replacement of the Rinehart cohomology $H^*_{\mathrm{Rin}}(\mathcal{L}, -)$ for general $\mathcal{L}$. The idea of the construction is very simple. We first observe that the transformation Lie–Rinehart algebras (see Example 2.2) are always free as A-modules, therefore the Rinehart cohomology of such algebras gives the correct answer. Secondly, for any Lie–Rinehart algebra $\mathcal{L}$ the two-sided bar construction $B_*(\mathrm{A}, \mathrm{A}, \mathcal{L})$ gives rise to a simplicial resolution of $\mathcal{L}$ in the category of Lie–Rinehart algebras. Since each term of this resolution is a transformation Lie–Rinehart algebra we can mix the Chevalley–Eilenberg complexes with the bar resolution to get our bicomplex.

### 3.1. A bicomplex for Lie–Rinehart algebras

Let $\mathcal{L}$ be a Lie–Rinehart algebra and let $M$ be a Lie–Rinehart module over $\mathcal{L}$. We have two cochain complexes: the Rinehart complex $C^*_{\mathrm{A}}(\mathcal{L}, M)$ and the Chevalley–Eilenberg complex $C^*_{\mathrm{Lie}}(\mathcal{L}, M)$. If one forgets the A-module structure on $\mathcal{L}$, we get a Lie $K$-algebra acting on A via derivations, thus the construction of Example 2.2 gives a Lie–Rinehart algebra structure on $\mathrm{A}\otimes\mathcal{L}$. We can iterate this construction to conclude that $\mathrm{A}^{\otimes n} \otimes \mathcal{L}$ is also a Lie–Rinehart algebra for any $n \geqslant 0$. The A-module structure comes from the first factor, while the bracket is a bit more complicated, for example for $n = 2$, we have

$$\begin{aligned}[a_1 \otimes a_2 \otimes X, b_1 \otimes b_2 \otimes Y] := {} & a_1b_1 \otimes a_2b_2 \otimes [X, Y] + a_1b_1 \otimes a_2X(b_2) \otimes Y \\ & + a_1a_2X(b_1) \otimes b_2 \otimes Y - a_1b_1 \otimes b_2Y(a_2) \otimes X \\ & - b_1b_2Y(a_1) \otimes a_2 \otimes X.\end{aligned}$$

Let us also recall that the two-sided bar construction $B_*(\mathrm{A}, \mathrm{A}, \mathcal{L})$ is a simplicial object, which is $\mathrm{A}^{\otimes n+1} \otimes \mathcal{L}$ in the dimension $n$, while the face maps are given by

$$d_i(a_0 \otimes \cdots \otimes a_n \otimes X) = a_0 \otimes \cdots \otimes a_ia_{i+1} \otimes \cdots \otimes a_n \otimes X,$$

if $i < n$ and

$$d_n(a_0 \otimes \cdots \otimes a_n \otimes X) = a_0 \otimes \cdots \otimes a_{n-1} \otimes a_nX,$$

if $i = n$. The degeneracy maps are given by

$$s_i(a_0 \otimes \cdots \otimes a_n \otimes X) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots \otimes a_n \otimes X.$$

In fact $B_*(\mathrm{A}, \mathrm{A}, \mathcal{L})$ is an augmented simplicial object in the category of Lie–Rinehart algebras, the augmentation $B_0(\mathrm{A}, \mathrm{A}, \mathcal{L}) = \mathrm{A} \otimes \mathcal{L} \to \mathcal{L}$ is given by $(a, X) \mapsto aX$. We can apply the functor $C^*_{\mathrm{A}}(-, M)$ on $B_*(\mathrm{A}, \mathrm{A}, \mathcal{L})$ to get a cosimplicial object in the category of cochain complexes

$$[n] \mapsto C^*_{\mathrm{A}}\big(\mathrm{A}^{\otimes n+1} \otimes \mathcal{L}, M\big).$$

Finally we let $C^{**}(\mathrm{A}, \mathcal{L}, M)$ be the bicomplex associated to this cosimplicial cochain complex. We let $H^*(\mathrm{A}, \mathcal{L}, M)$ be the cohomology of the corresponding total complex. The augmentation $B_*(\mathrm{A}, \mathrm{A}, \mathcal{L}) \to \mathcal{L}$ yields the homomorphism

$$\alpha^* : H^*_{\mathrm{Rin}}(\mathcal{L}, M) \to H^*(\mathrm{A}, \mathcal{L}, M).$$

The bicomplex $C^{**}(\mathrm{A}, \mathcal{L}, M)$ has the following alternative description. According to Lemma 2.9 we have the isomorphism of complexes:

$$C^{p*}(\mathrm{A}, \mathcal{L}, M) \cong C^*_{\mathrm{Lie}}\big(\mathrm{A}^{\otimes p} \otimes \mathcal{L}, M\big),$$

where $M$ is considered as a module over $\mathrm{A}^{\otimes p} \otimes \mathcal{L}$ by

$$(a_1 \otimes \cdots \otimes a_p \otimes X)m := (a_1 \cdots a_pX)m.$$

To define the horizontal cochain complex structure we observe that elements of $C^{pq}$ can be identified with functions $f : \mathrm{A}^{\otimes pq} \otimes \mathcal{L}^{\otimes q} \to M$, which are alternative with appropriate blocks of variables. Then the corresponding linear map

$$d(f) : \mathrm{A}^{\otimes (p+1)q} \otimes \mathcal{L}^{\otimes q} \to M$$

is given by

$$
\begin{aligned}
&df(a_{01},\dots,a_{0q},a_{11},\dots,a_{1q},\dots,a_{p1},\dots,a_{pq},X_1,\dots,X_q)\\
&\quad = a_{01}\cdots a_{0q} f(a_{11},\dots,a_{1q},\dots,a_{p1},\dots,a_{pq},X_1,\dots,X_q)\\
&\qquad + \sum_{0\leqslant i<p} (-1)^{i+1} f(a_{01},\dots,a_{0q},\dots,a_{i1}a_{i+1,1},\dots,a_{iq}a_{i+1,q},\dots,a_{p1},\dots,a_{pq},\\
&\qquad\qquad X_1,\dots,X_q)\\
&\qquad + (-1)^{p+1} f(a_{01},\dots,a_{0q},\dots,a_{p-1,1},\dots,a_{p-1,q},a_{p1}X_1,\dots,a_{pq}X_q).
\end{aligned}
$$

**Theorem 3.1.**

(i) *The homomorphism*

$$\alpha^n : H^n_{\mathrm{Rin}}(\mathcal{L}, M) \to H^n(\mathrm{A}, \mathcal{L}, M)$$

*is an isomorphism for $n = 0, 1$. The homomorphism $\alpha^2$ is a monomorphism. Moreover $\alpha^n$ is an isomorphism for all $n \geqslant 0$ provided $\mathcal{L}$ is projective over* A.

(ii) *If $0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence in the category* $(\mathcal{L}, \mathrm{A})$-**mod**, *then we have a long exact sequence on cohomology*

$$\cdots \to H^n(\mathrm{A}, \mathcal{L}, M_1) \to H^n(\mathrm{A}, \mathcal{L}, M) \to H^n(\mathrm{A}, \mathcal{L}, M_2) \to \cdots.$$

(iii) *The cohomology $H^2(\mathrm{A}, \mathcal{L}, M)$ classifies all abelian extensions*

$$0 \to M \to \mathcal{L}' \to \mathcal{L} \to 0$$

*of $\mathcal{L}$ by $M$ in the category of Lie–Rinehart algebras.*

(iv) *For any Lie–Rinehart algebra $\mathcal{L}$ and any Lie–Rinehart module $M$ there exists a natural bijection between the classes of the connected components of the category* **Cross**$(\mathcal{L}, M)$ *and* $H^3(\mathrm{A}, \mathcal{L}, M)$.

**Proof.** (i) The statement is obvious for $n = 0, 1$. For $n = 2$ it follows from part (iii) below and Theorem 2.8(iii). It remains to prove the last assertion. It is well known that the augmentation $B_*(\mathrm{A}, \mathrm{A}, \mathcal{L}) \to \mathcal{L}$ is a homotopy equivalence in the category of simplicial vector spaces, thanks to the existence of the extra degeneracy map given by $s(a_0 \otimes \cdots \otimes a_n \otimes X) = 1 \otimes a_0 \otimes \cdots \otimes a_n \otimes X$. However $s$ is not A-linear and therefore in general $B_*(\mathrm{A}, \mathrm{A}, \mathcal{L}) \to \mathcal{L}$ is only a weak equivalence in the category of simplicial A-modules. Assume now $\mathcal{L}$ is projective as an A-module, then $B_*(\mathrm{A}, \mathrm{A}, \mathcal{L}) \to \mathcal{L}$ is a homotopy equivalence in the category of simplicial A-modules and therefore, for each $k \geqslant 0$ the induced map $\Lambda^k_{\mathrm{A}}(B_*(\mathrm{A}, \mathrm{A}, \mathcal{L})) \to \Lambda^k_{\mathrm{A}}(\mathcal{L})$ is a homotopy equivalence in the category of simplicial A-modules, which implies that the same is true after applying the functor $\mathrm{Hom}_{\mathrm{A}}(-, M)$. Thus for each $k \geqslant 0$ the induced map $C^k_{\mathrm{A}}(\mathcal{L}, M) \to C^k_{\mathrm{A}}(B_*(\mathrm{A}, \mathrm{A}, \mathcal{L}))$ is a weak equivalence of cosimplicial objects and the comparison theorem for bicomplexes yields the result.

(ii) Since Hom and exterior powers involved in $C^m_{\mathrm{Lie}}(\mathfrak{g}, M)$ are taken over $K$ it follows that for each $p$ and $q$ the functor $C^q_{\mathrm{Lie}}(\mathrm{A}^p \otimes \mathcal{L}, -)$ is exact and the result follows.

(iii) Thanks to a well-known fact from topology we can use the normalized (in the simplicial direction) cochains to compute $H^*(\mathrm{A}, \mathcal{L}, M)$. Having this in mind we have $H^2(\mathrm{A}, \mathcal{L}, M) = Z^2/B^2$, where $Z^2$ consists of pairs $(f, g)$ such that $f : \Lambda^2(\mathcal{L}) \to M$ is a Lie 2-cocycle and $g : \mathrm{A} \otimes \mathcal{L} \to M$ is a linear map such that $g(1, X) = 0$,

$$ag(b, X) - g(ab, X) + g(a, bX) = 0$$

and

$$\begin{aligned} &abf(X, Y) - f(aX, bY) \\ &\quad = aXg(b, Y) - bYg(a, X) - g\bigl(ab, [X, Y]\bigr) - g\bigl(aX(b), Y\bigr) + g\bigl(bY(a), X\bigr). \end{aligned}$$

Here $a, b \in \mathrm{A}$ and $X, Y \in \mathcal{L}$. Moreover $(f, g)$ belongs to $B^2$ iff there exists a linear map $h : \mathcal{L} \to M$ such that $f(X, Y) = Xh(Y) - h([X, Y]) - Yh(X)$ and $g(a, X) = ah(X) - h(aX)$. Starting with $(f, g) \in Z^2$ we construct an abelian extension of $\mathcal{L}$ by $M$ by putting $\mathcal{P} = M \oplus \mathcal{L}$ as a vector space. An A-module structure on $\mathcal{P}$ is given by $a(m, X) = (am + g(a, X), aX)$, while a Lie bracket on $\mathcal{P}$ is given by $[(m, X), (n, Y)] = (X(n) - Y(m) + f(X, Y), [X, Y])$. Conversely, given an abelian extension $(\mathcal{P})$ and a $K$-linear section $h : \mathcal{L} \to \mathcal{P}$ we put $f(X, Y) := [h(X), h(Y)] - h([X, Y])$ and $g(a, X) := h(aX) - ah(X)$. It is easily checked that $(f, g) \in Z^2$ and we get (iii).

(iv) Similarly, we have $H^3(\mathrm{A}, \mathcal{L}, M) = Z^3/B^3$. Here $Z^3$ consists of triples $(f, g, h)$ such that $f : \Lambda^3(\mathcal{L}) \to M$ is a Lie 3-cocycle, $g : \Lambda^2(\mathrm{A} \otimes \mathcal{L}) \to M$ and $h : \mathrm{A} \otimes \mathrm{A} \otimes \mathcal{L} \to M$ are linear maps and the following relations hold:

$$\begin{aligned} &f(aX, bY, cZ) - abcf(X, Y, Z) \\ &\quad = aXg(b, c, Y, Z) - bYg(a, c, X, Z) + cZg(a, b, X, Y) - g\bigl(ab, c, [X, Y], Z\bigr) \\ &\qquad + g\bigl(aX(b), c, Y, Z\bigr) - g\bigl(bY(a), c, X, Z\bigr) + g\bigl(ac, b, [X, Y], Y\bigr) - g\bigl(aX(c), b, Z, Y\bigr) \\ &\qquad + g\bigl(cZ(a), b, X, Y\bigr) - g\bigl(bc, a, [Y, Z], X\bigr) + g\bigl(bY(c), a, Z, X\bigr) - g\bigl(cZ(b), a, Y, X\bigr) \end{aligned}$$

and

$$\begin{aligned} &abXh(c, d, Y) - cdYh(a, b, X) - h\bigl(ac, bd, [X, Y]\bigr) - h\bigl(ac, bX(d), Y\bigr) \\ &\quad - h\bigl(abX(c), d, Y\bigr) + h\bigl(ac, dY(b), X\bigr) - h\bigl(cdY(a), b, X\bigr) \\ &\quad = abg(c, d, X, Y) - g(ac, bd, X, Y) + g(a, b, cX, dY). \end{aligned}$$

Moreover $(f, g, h)$ belongs to $B^3$ iff there exist linear maps $m : \Lambda^2(\mathcal{L}) \to M$ and $n : \mathrm{A} \otimes \mathcal{L} \to M$ such that

$$f(X,Y,Z) = Xm(Y,Z) - Ym(X,Z) + Zm(X,Y) - m\big([X,Y],Z\big) + m\big([X,Z],Y\big) \\ - m\big([Y,Z],X\big),$$
$$g(a,b,X,Y) = abm(X,Y) - m(aX,bY) - aXn(b,Y) + bYn(a,X) + n\big(ab,[X,Y]\big) \\ + n\big(aX(b),Y\big) + n\big(bY(a),X\big)$$

and

$$h(a,b,X) = an(b,X) - n(ab,X) + n(a,bX).$$

Let

$$0 \to M \to \mathcal{R} \xrightarrow{\partial} \mathcal{P} \xrightarrow{\pi} \mathcal{L} \to 0$$

be a crossed extension. We put $V := \mathsf{Im}(\partial)$ and consider $K$-linear sections $p : \mathcal{L} \to \mathcal{P}$ and $q : V \to \mathcal{R}$ of $\pi : \mathcal{P} \to \mathcal{L}$ and $\partial : \mathcal{R} \to V$ respectively. Now we define $t : \mathcal{L} \otimes \mathcal{L} \to \mathcal{R}$ and $s : \mathrm{A} \otimes \mathcal{L} \to \mathcal{R}$ by $t(X,Y) := q([p(X), p(Y)] - p([X,Y]))$ and $s(a,X) := q(ap(X) - p(aX))$. Finally we define three functions as follows. The function $f : \Lambda^3(\mathcal{L}) \to M$ is given by

$$f(X,Y,Z) := p(X)g(Y,Z) - p(Y)g(X,Z) + p(Z)g(X,Y) - g\big([X,Y],Z\big) \\ + g\big([X,Z],Y\big) - g\big([Y,Z],X\big).$$

The function $g : \Lambda^2(\mathrm{A} \otimes \mathcal{L}) \to M$ is given by

$$g(a,b,X,Y) := p(aX)s(b,Y) - p(bY)s(a,X) - p\big(ab,[X,Y]\big) - p\big(aX(b),Y\big) \\ + p\big(bY(a),X\big) - t(aX,bY) + abt(X,Y),$$

while the function $h : \mathrm{A} \otimes \mathrm{A} \otimes \mathcal{L} \to M$ is given by

$$h(a,b,X) := as(b,X) - s(ab,X) + s(a,bX).$$

Then $(f,g,h) \in Z^3$ and the corresponding class in $H^3(\mathrm{A},\mathcal{L},M)$ depends only on the connected component of a given crossed extension. Thus we obtain a well-defined map $\mathbf{Cross}(\mathcal{L},M) \to H^3(\mathrm{A},\mathcal{L},M)$ and a standard argument (see [8]) shows that it is an isomorphism. $\square$

## 4. Sridharan representations of Lie–Rinehart algebras

In this section we extend the definition of Lie–Rinehart module in the spirit of the classical work of Sridharan [15].

Let $\mathcal{L}$ be a Lie–Rinehart A-algebra and let $f : \mathcal{L} \otimes \mathcal{L} \to \mathrm{A}$ and $g : \mathrm{A} \otimes \mathcal{L} \to \mathrm{A}$ be linear maps. A *Sridharan module* is an A-module $M$ together with a $K$-linear map

$$\mathcal{L} \otimes M \to M, \quad (X, m) \mapsto X(m),$$

such that the following identities hold:

(i) $[X, Y](m) + f(X, Y)m = X(Y(m)) - Y(X(m))$,
(ii) $X(am) + g(a, X)m = aX(m) + X(a)(m)$,
(iii) $(aX)(m) = a(X(m))$.

The proof of the following lemma is a straightforward and simple computation similar to [15, Proposition 1.2] and therefore we omit it.

**Lemma 4.1.** *If $M$ is a Sridharan module such that $M$ is faithful as an* A*-module, then $(f, g)$ defines a normalized* 2*-cocycle in the total complex of the bicomplex $C^{**}(\mathrm{A}, \mathcal{L}, \mathrm{A})$. Thus $f : \Lambda^2(\mathcal{L}) \to \mathrm{A}$ is a Lie* 2*-cocycle and the following identities hold*:

$$\begin{aligned} g(1, X) &= 0, \\ ag(b, X) - g(ab, X) + g(a, bX) &= 0, \\ abf(X, Y) - f(aX, bY) &= aXg(b, Y) - bYg(a, X) - g\big(ab, [X, Y]\big) \\ &\quad - g\big(aX(b), Y\big) + g\big(bY(a), X\big). \end{aligned}$$

In what follows we will assume that the pair $(f, g)$ is a normalized 2-cocycle in the total complex of the bicomplex $C^{**}(\mathrm{A}, \mathcal{L}, \mathrm{A})$. There is a $K$-algebra $V(\mathrm{A}, \mathcal{L}, f, g)$ with properties such that the category of $V(\mathrm{A}, \mathcal{L}, f, g)$-modules is isomorphic to the category of Sridharan representations. Actually this algebra for $\mathrm{A} = K$ and $g = 0$ was constructed in [15], while for arbitrary A but $f = 0 = g$ it appears in [14]. We define the algebra $V(\mathrm{A}, \mathcal{L}, f, g)$ in terms of generators and relations. We have generators $i(X)$ for each $X \in \mathcal{L}$ and $j(a)$ for each $a \in \mathrm{A}$. These generators must satisfy the following relations:

$$\begin{aligned} j(1) = 1, \quad j(ab) &= j(a)j(b), \\ i(aX) &= j(a)i(X), \\ i(X)i(Y) - i(Y)i(X) &= i\big([X, Y]\big) + j\big(f(X, Y)\big), \\ i(X)j(a) &= j(a)i(X) + j\big(X(a) - g(a, X)\big). \end{aligned}$$

The first relations show that $j : \mathrm{A} \to V(\mathrm{A}, \mathcal{L}, f, g)$ is an algebra homomorphism. We let $V_n$ be the A-submodule spanned on all products $i(X_1) \cdots i(X_k)$, where $k \leqslant n$. Then

$$0 \subset \mathrm{A} = V_0 \subset V_1 \subset \cdots \subset V_n \subset \cdots \subset V(\mathrm{A}, \mathcal{L}, f, g)$$

defines an algebra filtration on $V(\mathrm{A}, \mathcal{L}, f, g)$. It is clear that $V(\mathrm{A}, \mathcal{L}, f, g) = \bigcup_{n \geqslant 0} V_n$. It follows from the third relation that the associated graded object $\mathsf{gr}_*(V)$ is a commutative A-algebra. In other words $V(\mathrm{A}, \mathcal{L}, f, g)$ is an almost commutative algebra in the following sense.

An *almost commutative algebra* is an associative $K$-algebra $C$ together with a filtration

$$0 \subset \mathrm{A} = C_0 \subset C_1 \subset \cdots \subset C_n \subset \cdots \subset C = \bigcup_{n \geqslant 0} C_n$$

such that $C_n C_m \subset C_{n+m}$ and such that the associated graded object

$$\mathsf{gr}_*(C) = \bigoplus_{n \geqslant 0} C_n / C_{n-1}$$

is a commutative A-algebra. It is well known that if $C$ is an almost commutative algebra, then there is a well-defined bracket

$$[-,-] : \mathsf{gr}_n(C) \otimes \mathsf{gr}_m(C) \to \mathsf{gr}_{n+m-1}(C)$$

which is given as follows. Let $a \in \mathsf{gr}_n(C)$ and $b \in \mathsf{gr}_m(C)$ and $\hat{a} \in C_n$ and $\hat{b} \in C_m$ representing $a$ and $b$ respectively. Since $\mathsf{gr}_*(C)$ is a commutative algebra it follows that $\hat{a}\hat{b} - \hat{b}\hat{a} \in C_{n+m-1}$ and the corresponding class in $\mathsf{gr}_{n+m-1}(C)$ is $[a, b]$. It is also well known that in this way we obtain a Poisson algebra structure on $\mathsf{gr}_*(C)$. Since the bracket is of degree $(-1)$ it follows from Example 2.3 that $\mathcal{L} = \mathsf{gr}_1(C)$ is a Lie–Rinehart algebra over $\mathrm{A} = \mathsf{gr}_0(C)$. Moreover the exact sequence

$$0 \to \mathrm{A} \to C_1 \to \mathcal{L} \to 0$$

is an abelian extension of Lie–Rinehart algebras and therefore any $K$-linear section of the projection $C_1 \to \mathcal{L}$ defines a 2-cocycle $(f, g)$ of $C^{**}(\mathrm{A}, \mathcal{L}, \mathrm{A})$ and the homomorphism of associative algebras

$$V(\mathrm{A}, \mathcal{L}, f, g) \to C.$$

Using a similar argument as in [15] we prove that this map is an isomorphism provided $\mathcal{L}$ is free as an A-module and the natural map $S^*_{\mathrm{A}}(\mathcal{L}) \to \mathsf{gr}_*(C)$ is an isomorphism. Here $S^*$ denotes the symmetric algebra.

## 5. Triple cohomology of Lie–Rinehart algebras

In this section we prove that the cohomology theory developed in the previous section is canonically isomorphic to the triple cohomology of Barr–Beck [1] applied to Lie–Rinehart algebras.

### 5.1. Cotriples and cotriple resolutions

The general notions of (co)triples (or (co)monads, or (co)standard construction) and (co)triple resolutions are due to Godement [5] and further developed in [1]. Let $\mathcal{C}$ be a category. A *cotriple* on $\mathcal{C}$ is an endofunctor $T:\mathcal{C}\to\mathcal{C}$ together with natural transformations $\varepsilon:T\to 1_{\mathcal{C}}$ and $\delta:T\to T^2$ satisfying the counit and the coassociativity properties. Here $T^2=T\circ T$ and a similar meaning has $T^n$ for all $n\geqslant 0$. For example, assume $U:\mathcal{C}\to\mathcal{B}$ is a functor which has a left adjoint functor $F:\mathcal{B}\to\mathcal{C}$. Then there is a cotriple structure on $T=FU:\mathcal{C}\to\mathcal{C}$ such that $\varepsilon$ is the counit of the adjunction. Given a cotriple $T$ and an object $C$, a simplicial object $T_*C$ in the category $\mathcal{C}$, known as *Godement or cotriple resolution of $C$*, can be associated. Let us recall that $T_nC=T^{n+1}C$ and the face and degeneracy operators are given respectively by $\partial_i=T^i\varepsilon T^{n-i}$ and $s_i=T^i\delta T^{n-i}$. To explain why it is called resolution, consider the case when $T=FU$ is associated to the pair of adjoint functors. Then firstly $\varepsilon$ yields a morphism $T_*C\to C$ from the simplicial object $T_*C$ to the constant simplicial object $C$ and secondly the induced morphism $U(T_*C)\to U(C)$ is a homotopy equivalence in the category of simplicial objects in $\mathcal{B}$. The cotriple cohomology is now defined as follows. Let $M$ be an abelian group object in the category $\mathcal{C}/C$ of arrows $X\to C$ then $\mathrm{Hom}_{\mathcal{C}/C}(T_*C,M)$ is a cosimplicial abelian group, which can also be seen as a cochain complex. Thus $H^*(\mathrm{Hom}_{\mathcal{C}/C}(T_*C,M))$ are meaningful and they are denoted by $H^*_T(C,M)$. Of special interest is the case, when $T=FU$ is associated to the pair of adjoint functors and the functor $U:\mathcal{C}\to\mathcal{B}$ is *tripleable* [1]. In this case the category $\mathcal{C}$ is completely determined by the triple $E=UF:\mathcal{B}\to\mathcal{B}$. Because of this fact, in this case $H^*_T(C,M)$ are known as *triple cohomology of $C$ with coefficients in $M$*.

### 5.2. Free Lie–Rinehart algebras

We wish to apply these general constructions to Lie–Rinehart algebras. We have the functor

$$U:\mathcal{LR}(\mathrm{A})\to \mathrm{Vect}/\mathrm{Der}(\mathrm{A})$$

which assigns $\alpha:\mathcal{L}\to\mathrm{Der}(\mathrm{A})$ to a Lie–Rinehart algebra $\mathcal{L}$. Here $\mathrm{Vect}/\mathrm{Der}(\mathrm{A})$ is the category of $K$-linear maps $\psi:V\to\mathrm{Der}(\mathrm{A})$, where $V$ is a vector space over $K$. A morphism $\psi\to\psi_1$ in $\mathrm{Vect}/\mathrm{Der}(\mathrm{A})$ is a $K$-linear map $f:V\to V_1$ such that $\psi=\psi_1\circ f$. Now we construct the functor

$$F:\mathrm{Vect}/\mathrm{Der}(\mathrm{A})\to\mathcal{LR}(\mathrm{A})$$

as follows. Let $\psi:V\to\mathrm{Der}(\mathrm{A})$ be a $K$-linear map. We let $\mathbf{L}(V)$ be the free Lie $K$-algebra generated by $V$. Then we have the unique Lie $K$-algebra homomorphism $\mathbf{L}(V)\to\mathrm{Der}(\mathrm{A})$ which extends the map $\psi$, which is still denoted by $\psi$. Now we can apply the construction from Example 2.2 to get a Lie–Rinehart algebra structure on $\mathrm{A}\otimes\mathbf{L}(V)$. We let $F(\psi)$ be this particular Lie–Rinehart algebra and we call it the *free Lie–Rinehart algebra generated by $\psi$*. In this way we obtain the functor $F$, which is the left adjoint to $U$.

**Lemma 5.1.** *Let $\mathcal{L}$ be a free Lie–Rinehart algebra generated by $\psi : V \to \mathrm{Der}(\mathrm{A})$ and let $M$ be any Lie–Rinehart module over $\mathcal{L}$. Then*

$$H^i_{\mathrm{Rin}}(\mathcal{L}, M) = 0, \quad i > 1.$$

**Proof.** By our construction $\mathcal{L}$ is a transformation Lie–Rinehart algebra of $(\mathbf{L}(V), \mathrm{A})$. Thus we can apply Lemma 2.9 to get an isomorphism $H^*_{\mathrm{Rin}}(\mathcal{L}, M) \cong H^*_{\mathrm{Lie}}(\mathbf{L}(V), M)$ and then we can use the well-known vanishing result for free Lie algebras. □

*5.3. The cohomology $H^*_{\mathrm{LR}}(\mathcal{L}, M)$*

Since we have a pair of adjoint functors we can take the composite

$$T = FU : \mathcal{LR}(\mathrm{A}) \to \mathcal{LR}(\mathrm{A})$$

which is a cotriple. Thus for any Lie–Rinehart algebra $\mathcal{L}$ we can take the cotriple resolution $T_*(\mathcal{L}) \to \mathcal{L}$. It follows from the construction of the cotriple resolution that each component of $T_*(\mathcal{L})$ is a free Lie–Rinehart algebra. Moreover according to the general properties of the cotriple resolutions the natural augmentation $T_*(\mathcal{L}) \to \mathcal{L}$ is a homotopy equivalence in the category of simplicial vector spaces. It follows that $T_*(\mathcal{L}) \to \mathcal{L}$ is a weak homotopy equivalence in the category of A-modules.

Let $M$ be an $\mathcal{L}$-module. Then $M$ is also a module over $T_n(\mathcal{L})$ for any $n \geqslant 0$ thanks to the augmentation morphism $T_*(\mathcal{L}) \to \mathcal{L}$. Thus we can form the following bicomplex

$$C^*_{\mathrm{A}}\big(T_*(\mathcal{L}), M\big)$$

which is formed by the degreewise applying the Rinehart cochain complex. The cohomology of the total complex of the bicomplex $C^*_{\mathrm{A}}(T_*(\mathcal{L}), M)$ is denoted by $H^*_{\mathrm{LR}}(\mathcal{L}, M)$.

**Lemma 5.2.** *For any Lie–Rinehart algebra $\mathcal{L}$ and any Lie–Rinehart module $M$ we have a natural isomorphism*

$$H^*(\mathrm{A}, \mathcal{L}, M) \cong H^*_{\mathrm{LR}}(\mathcal{L}, M).$$

**Proof.** We denote by $C^*(\mathrm{A}, \mathcal{L}, M)$ the total complex associated to the bicomplex $C^{**}(\mathrm{A}, \mathcal{L}, M)$. Recall that it comes with a natural cochain map

$$C^*_{\mathrm{A}}(\mathcal{L}, M) \to C^*(\mathrm{A}, \mathcal{L}, M)$$

which is a quasi-isomorphism provided $\mathcal{L}$ is projective as an A-module. Let us apply $C^*(\mathrm{A}, -, M)$ on $T_*(\mathcal{L})$ degreewise. Then we obtain the morphism of bicomplex

$$C^*_{\mathrm{A}}\big(T_*(\mathcal{L}), M\big) \to C^*\big(\mathrm{A}, T_*(\mathcal{L}), M\big)$$

which is a quasi-isomorphism because each $T_n(\mathcal{L})$ is free as an A-module. It remains to show that the augmentation $T_*(\mathcal{L}) \to \mathcal{L}$ yields the quasi-isomorphism

$$C^*(\mathrm{A}, \mathcal{L}, M) \to C^*\big(\mathrm{A}, T_*(\mathcal{L}), M\big).$$

To this end, we observe that $T_*(\mathcal{L}) \to \mathcal{L}$ is a quasi-isomorphism thanks to the general properties of cotriple resolutions and therefore is a homotopy equivalence in the category of simplicial vector spaces. Thus the same is true for $\Lambda^n(T_*(\mathcal{L})) \to \Lambda^n(\mathcal{L})$ and therefore $C^n(\mathrm{A}, \mathcal{L}, M) \to C^n(\mathrm{A}, T_*(\mathcal{L}), M)$ is also a homotopy-equivalence for each $n$ and the result follows from the comparison theorem of bicomplexes. $\square$

### 5.4. *Triple cohomology and $H^*_{\mathrm{LR}}(\mathcal{L}, M)$*

According to the Beck's tripleability criterion the functor $U : \mathcal{LR}(\mathrm{A}) \to \mathrm{Vect}/\mathrm{Der}(\mathrm{A})$ is tripleable, so we also have the triple cohomology theory for Lie–Rinehart algebras. Let $\mathcal{L}$ be a Lie–Rinehart algebra. There is an equivalence from the category of Lie–Rinehart modules over $\mathcal{L}$ to the category of abelian group objects in $\mathcal{LR}(\mathrm{A})/\mathcal{L}$, which assigns the projection $\mathcal{L} \rtimes M \to \mathcal{L}$ to $M \in (\mathcal{L}, \mathrm{A})$-**mod**. Having this equivalence in mind, Lemma 2.5 says that for any object $\mathcal{P} \to \mathcal{L}$ of $\mathcal{LR}(\mathrm{A})/\mathcal{L}$ the homomorphisms from $\mathcal{P} \to \mathcal{L}$ to $\mathcal{L} \rtimes M \to \mathcal{L}$ in the category of abelian group objects in $\mathcal{LR}(\mathrm{A})/\mathcal{L}$ is nothing but $\mathrm{Der}_{\mathrm{A}}(\mathcal{P}, M)$. Therefore the triple cohomology $H^*_T(\mathcal{L}, M)$ is the same as $H^q(\mathrm{Der}_{\mathrm{A}}(T_*(\mathcal{L}), M))$.

**Theorem 5.3.** *For any Lie–Rinehart algebra $\mathcal{L}$ and any $\mathcal{L}$-module $M$ there is a natural isomorphism*:

$$H^{q+1}_{\mathrm{LR}}(\mathcal{L}, M) \cong H^q_T(\mathcal{L}, M), \quad q > 0.$$

*In other words the cotriple cohomology of $\mathcal{L}$ with coefficients in $M$ is isomorphic to the cohomology $H^*_{\mathrm{LR}}(\mathcal{L}, M)$ up to shift in the dimension.*

**Proof.** As usual with bicomplex we have a spectral sequence

$$E^2_{pq} \Rightarrow H^*_{\mathrm{LR}}(\mathcal{L}, M)$$

where $E^2_{pq}$ is obtained in two steps: We first take $p$th homology in each $C^*(T_q(\mathcal{L}), M)$, $q \geqslant 0$ and then we take the $q$th homology. But $C^*(T_q(\mathcal{L}), M)$ is just the Rinehart complex of $T_q(\mathcal{L})$. Since $T_q(\mathcal{L})$ is free we can use Lemma 5.1 to conclude that $E^1_{pq} = 0$ for all $p \geqslant 2$. According to the exact sequence (1) we also have an exact sequence

$$0 \to E^1_{0q} \to M \to \mathrm{Der}_{\mathrm{A}}\big(T_q(\mathcal{L}), M\big) \to E^1_{1q} \to 0.$$

We observe that $E^1_{0*}$ and $M$ are constant cosimplicial vector spaces and therefore $E^2_{0q} = 0$ for all $q > 0$. Thus we get

$$H^{q+1}_{\mathrm{LR}}(\mathcal{L}, M) \cong E^2_{1q} \cong H^q\big(\mathrm{Der}_{\mathrm{A}}\big(T_*(\mathcal{L}), M\big)\big), \quad q > 0. \qquad \square$$

## 6. The canonical class of associative algebras

Let $S$ be an associative algebra over $K$. We let A be the center of $S$. As an application of our results we construct a canonical class $o(S) \in H^3(\mathrm{A}, H^1(S,S), \mathrm{A})$, where $H^*(S,S)$ denotes the Hochschild cohomology of $S$.

Let us first recall the definitions of the zeroth and the first dimensional Hochschild cohomology involved in this construction. Let $S$ be an associative $K$-algebra. A *$K$-derivation* $D : S \to S$ is a $K$-linear map, such that $D(ab) = D(a)b + aD(b)$. We let $\mathrm{Der}(S)$ be the set of all $K$-derivations. It has a natural Lie $K$-algebra structure, where the bracket is defined via the commutator $[D, D_1] = DD_1 - D_1D$. There is a canonical $K$-linear map

$$\mathrm{ad} : S \to \mathrm{Der}(S)$$

given by $\mathrm{ad}(s)(x) = sx - xs$, $s, x \in S$. Then the zeroth and the first dimensional Hochschild cohomology groups are defined via the exact sequence:

$$0 \to H^0(S,S) \to S \stackrel{\mathrm{ad}}{\to} \mathrm{Der}(S) \to H^1(S,S) \to 0. \tag{2}$$

It follows that $\mathrm{A} = H^0(S,S)$ is the center of $S$. We claim that $\mathrm{Der}(S)$ is a Lie–Rinehart algebra over A. Indeed, the action of A is defined by $(aD)(s) = aD(s)$, $D \in \mathrm{Der}(S)$, $s \in S$, $a \in \mathrm{A}$, while the homomorphism $\alpha : \mathrm{Der}(S) \to \mathrm{Der}(\mathrm{A})$ is just the restriction. To see that $\alpha$ is well defined, it suffices to show that $D(\mathrm{A}) \subset \mathrm{A}$ for any $D \in \mathrm{Der}(S)$. To this end, let us observe that for any $s \in S$ and $a \in \mathrm{A}$ we have

$$D(a)s - sD(a) = \big(D(as) - aD(s)\big) - \big(D(sa) - D(s)a\big) = 0$$

and therefore $D(a) \in \mathrm{A}$. On the other hand the commutator $[s,t] = st - ts$ defines a Lie A-algebra structure on $S$ and $\mathrm{ad} : S \to \mathrm{Der}(S)$ is a Lie $K$-algebra homomorphism. Actually more is true: ad is a crossed module of Lie–Rinehart algebras over A, where the action of the Lie–Rinehart algebra $\mathrm{Der}(S)$ on $S$ is given by $(D, s) \mapsto D(s)$. It follows that $H^1(S,S) = \mathsf{Coker}(\mathrm{ad} : S \to \mathrm{Der}(S))$ is also a Lie–Rinehart algebra over A and $\mathrm{A} = \mathsf{Ker}(\mathrm{ad} : S \to \mathrm{Der}(S))$ is a Lie–Rinehart module over $H^1(S,S)$. In particular the groups $H^*(\mathrm{A}, H^1(S,S), \mathrm{A})$ are well defined. According to Theorem 3.1 the vector space $H^3(\mathrm{A}, H^1(S,S), \mathrm{A})$ classifies the crossed extension of $H^1(S,S)$ by A. By our construction the exact sequence (2) is one of such extension and therefore it defines a canonical class $o(S) \in H^3(\mathrm{A}, H^1(S,S), \mathrm{A})$. Since for a commutative algebra A the class $o(\mathrm{A})$ vanishes, one can think on it as a measure of noncommutativity of $S$.

**Lemma 6.1.** *$o(S)$ is a Morita invariant.*

**Proof.** Let $R$ be the $K$-algebra of $n \times n$ matrices. We have to prove that $o(S) = o(R)$. Let $D$ be a derivation of $S$. We let $g(D)$ be the derivation of $R$ which is componentwise extension of $D$. Furthermore, for an element $s \in S$ we let $f(s)$ be the diagonal matrix with $s$ on diagonals. Then we have the following commutative diagram

$$\begin{tikzcd} S \arrow[r, "\mathrm{ad}"] \arrow[d, "f"'] & \mathrm{Der}(S) \arrow[d, "g"] \\ R \arrow[r, "\mathrm{ad}"] & \mathrm{Der}(R) \end{tikzcd}$$

in the category $\mathcal{LR}(\mathrm{A})$ and the result follows from the fact that Hochschild cohomology is a Morita invariant. $\square$

Let us observe that if $S$ is a smooth commutative algebra, then $\mathrm{A} = S$ and $H^3(\mathrm{A}, H^1(S, S), \mathrm{A})$ is isomorphic to the de Rham cohomology of $S$ (of course $o(S) = 0$ in this case). So, in general we can consider the groups $H^3(\mathrm{A}, H^1(S, S), \mathrm{A})$ as a sort of noncommutative de Rham cohomology.

By forgetting the A-module structure, we obtain an element

$$o'(S) \in H^3_{\mathrm{Lie}}\big(H^1(S, S), \mathrm{A}\big).$$

These groups and probably the corresponding elements can be computed in many cases using the results of Strametz [16].

**Remark 6.2.**

(i) For any associative algebra $S$ there is a multiplicative version of the class $o(S)$, which corresponds to the crossed extension of groups

$$0 \to \mathsf{U}(\mathrm{A}) \to \mathsf{U}(S) \xrightarrow{\alpha} \mathsf{Aut}(S) \to \mathsf{Out}(S) \to 0.$$

Here as above A is the center of $S$, while $\mathsf{U}(S)$ is the group of invertible elements of $S$. Moreover $\mathsf{Aut}(S)$ is the group of algebra automorphisms of $S$ and $\alpha$ is given by $\alpha(t)(s) = t^{-1}st$, $s \in S$, $t \in \mathsf{U}(S)$. Thanks to [10] this extension defines an element in $H^3(\mathsf{Out}(S), \mathsf{U}(\mathrm{A}))$. Here $H^*$ denotes the cohomology of groups.

(ii) For any Poisson algebra $P$ there is a similar class $o(P)$, which corresponds to the following crossed extension of Lie–Rinehart algebras over $H^0_{\mathrm{Poiss}}(P, P)$:

$$0 \to H^0_{\mathrm{Poiss}}(P, P) \to P \xrightarrow{\mathsf{ad}} \mathsf{Der}_{\mathrm{Poiss}}(P) \to H^1_{\mathrm{Poiss}}(P, P) \to 0$$

where $\mathsf{ad}$ is given by $\mathsf{ad}(a) = [a, -]$ and $H^1_{\mathrm{Poiss}}(P, P)$ is just the cokernel of $\mathsf{ad}$. Since $\mathsf{ad} : P \to \mathsf{Der}_{\mathrm{Poiss}}(P)$ is a crossed module of Lie–Rinehart algebras over $H^0_{\mathrm{Poiss}}(P, P)$ it follows that $H^1_{\mathrm{Poiss}}(P, P)$ is also a Lie–Rinehart algebra over $H^0_{\mathrm{Poiss}}(P, P)$ and we get that the class $o(P)$ lies in $H^3(\mathrm{A}, H^1_{\mathrm{Poiss}}(P, P), \mathrm{A})$, where $\mathrm{A} = H^0_{\mathrm{Poiss}}(P, P)$.

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